SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class: Circle

A B 1 2 S

2023

YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- Unless otherwise stated, all answers should be left in simplified exact form
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 2 – 5)

Attempt Questions 1 – 10
Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 15)

Attempt all Questions in Section II
Allow about 2 hours and 45 minutes for this section

Examiner:
External Examiner

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section.
Use the multiple-choice answer sheet for Questions 1–10.

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1 A cubic equation with integer coefficients has $2 - 3i$ and 4 as two of its roots.

What is the third root?

A. $2 + 3i$ B. $3 - 2i$ C. $3 + 2i$ D. $-2 + 3i$

2 Which expression below is equivalent to

$$\frac{e^{-\frac{5i\pi}{6}}}{e^{\frac{i\pi}{2}}}\ ?$$

A. $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

B. $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

C. $-\frac{\sqrt{3}}{2}-\frac{1}{2}i$

D. $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$

3 The indefinite integral

$$\int \frac{a-3x}{4-x^2}\,dx$$

can be determined using the partial fraction decomposition

$$\frac{2}{2+x}-\frac{1}{2-x}.$$

What is the value of a ?

A. -2

B. -1

C. 2

D. 3

4 A particle is moving along the x-axis, initially moving to the left from the origin.
Its velocity and acceleration are given by

$$v^2 = 2\ln(3+\cos x) \text{ and } a = -\frac{\sin x}{3+\cos x}.$$

Which of the following describes its subsequent motion?

A. It moves only to the left, alternately speeding up and slowing down, without stopping.

B. It moves only to the left, alternately slowing to a stop and then speeding up.

C. It slows to a stop, then heads to the right forever.

D. It oscillates between two points

5 A particle of mass m kg is projected vertically upward with a velocity $v\,\mathbf{j}$ m/s.

The magnitude of the air resistance is given by $\dfrac{mgv^2}{\lambda^2}$, where λ is a constant.

Which expression describes the acceleration for the upward motion of the particle?

A. $-\dfrac{g}{\lambda^2}\left(\lambda^2+v^2\right)\mathbf{j}$

B. $-\dfrac{mg}{\lambda^2}\left(\lambda^2+v^2\right)\mathbf{j}$

C. $\dfrac{g}{\lambda^2}\left(\lambda^2-v^2\right)\mathbf{j}$

D. $\dfrac{mg}{\lambda^2}\left(\lambda^2-v^2\right)\mathbf{j}$

6 The velocity of a body moving in a straight line is given by $v = f(x)$,
where x metres is the distance from origin and v is the velocity in metres per second.
What is the acceleration of the body in m/s^2 ?

A. $f'(x)$

B. $f'(v)$

C. $x f'(x)$

D. $f(x) f'(x)$

7 A particle is moving in a straight line such that its velocity, in metres per second, is given by

$$v^2 = 20 - 16x - 4x^2,$$

where x is the displacement, in metres, of the particle from a fixed point, O.

Which of the following statements about the particle is true?

A. The particle moves in simple harmonic motion, oscillating about the centre $x = -2$ with a period of π and an amplitude of 3 metres.

B. The particle moves in simple harmonic motion, oscillating about the centre $x = -2$ with a period of $\frac{\pi}{2}$ and an amplitude of 3 metres.

C. The particle moves in simple harmonic motion, oscillating about the centre $x = 2$ with a period of π and an amplitude of 3 metres.

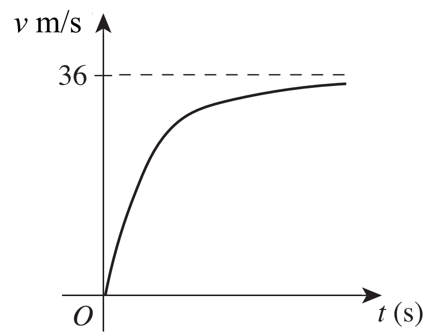
D. The particle moves in simple harmonic motion, oscillating about the centre $x = 2$ with a period of $\frac{\pi}{2}$ and an amplitude of 3 metres.

8 A mass of 1 kg is dropped from a height in a resistive medium under a constant gravitational acceleration of 10 m/s^2.

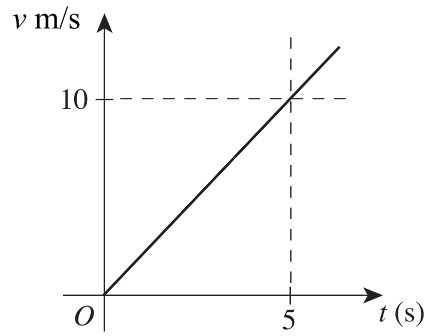
The resistive force is directly proportional to the speed v m/s.

If the constant of proportionality is 0.5, which of the following best represents the velocity-time graph of the mass?

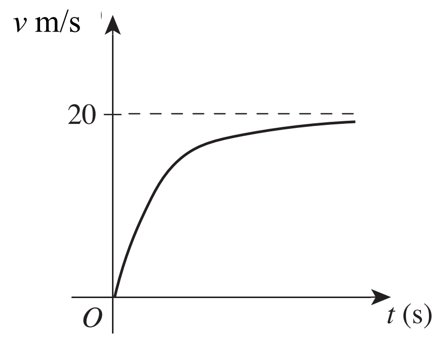
A.



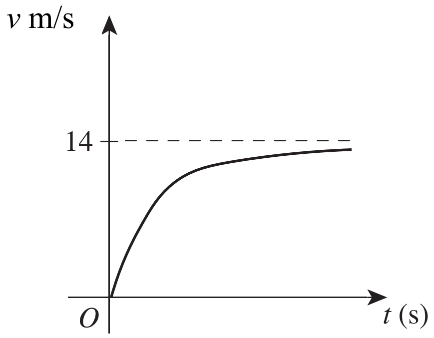
B.



C.



D.



9 What is the value of $\int_{a}^{a+b} \frac{g(2a+b-x)}{g(2a+b-x)-g(x)}\,dx$?

A. $\frac{a}{2}$

B. $\frac{b}{2}$

C. a

D. b

10 Given that ϕ is a complex number such that $\text{Re}(\phi) > \text{Im}(\phi) > 1$.

Which of the following can be true?

A. $|\phi| = \sqrt{2\,\text{Re}(\phi)\,\text{Im}(\phi)}$

B. $\left|\frac{\phi-\overline{\phi}}{\phi+\overline{\phi}}\right| > 1$

C. $\left|\text{Im}\left(\frac{\phi}{i}\right)\right| < \text{Im}(\phi)$

D. $|\phi| < \text{Re}(\phi) + \text{Im}(\phi)$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

In Questions 11-16, your responses should include ALL relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE writing booklet

(a) Consider complex numbers u and v, where $u = 1 + 2i$ and $v = 2 + i$. **3**

If $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{6\sqrt{2}}{w}$, find w in the form $a + ib$, where a and b are real numbers.

(b) The complex numbers z and w have moduli k and $3k^2$ respectively, where $k \in \mathbb{R}^+$.

Their arguments are α and 4α respectively, where $-\dfrac{\pi}{7} < \alpha \leq \dfrac{\pi}{7}$.

(i) Express $\dfrac{z^3}{\overline{w}}$ in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. **2**

(ii) It is given that $\alpha = \dfrac{\pi}{21}$.

Find the integer values of n such that $\operatorname{Im}\left(\left(\dfrac{z^3}{\overline{w}}\right)^n\right) = 0$. **2**

(c) (i) Differentiate $e^{\cos 2x}$ with respect to x. **1**

(ii) Show that **3**

$$\int e^{\cos 2x} \sin 4x \, dx = e^{\cos 2x}(1 - \cos 2x) + C.$$

(iii) Hence, find $\displaystyle\int e^{\cos 2x}(\sin x \cos 3x)\, dx$. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Evaluate **3**

$$\int_1^9 (x-1)\sqrt{16-(x-5)^2}\,dx.$$

(b) Find $\displaystyle\int \frac{1+\ln x}{x(2+\ln x)(3+\ln x)}\,dx$ **4**

(c) The complex numbers z_1 and z_2 are such that

$$z_1 = 1+bi,\ b>1,\ \text{Arg}(z_1)=\alpha \text{ and}$$

$$z_2 = 1-ci,\ 0<c<1,\ \text{Arg}(z_2)=\beta$$

Let Z_1 and Z_2 be points representing z_1 and z_2 respectively on the Argand diagram.

(i) Indicate Z_1 and Z_2 on an Argand diagram. **2**

Z_3 is the point representing z_3 on an Argand diagram such that $|z_3|=|z_1|$, the origin O is collinear with Z_2 and Z_3, and $z_3 = e^{ki}z_1$, where k is a real constant.

(ii) Indicate the two possible positions of Z_3 on the same Argand diagram drawn in part (i). **2**

(iii) Hence determine the two possible values of k, leaving your answers in terms of α and β. **2**

(iv) For the value of $k=\dfrac{\pi}{2}$, express the area of triangle $Z_1Z_2Z_3$, in terms of $|z_1|$ and $|z_2|$. **2**

Question 13 (16 marks) Use a SEPARATE writing booklet

(a) A particle is initially at $x = 1$ with a velocity of 2 m/s.
The acceleration of the particle is given by

$$a = \frac{1}{2}\left(1 - \frac{1}{x^2}\right) \text{ m/s}^2,$$

where x is the displacement of the particle from O.

(i) Prove that $\dfrac{dx}{dt} = \dfrac{1+x}{\sqrt{x}}$. **3**

(ii) Show that the time, in seconds, taken for the particle to reach $x = 3$ is **4**

$$2\left(\sqrt{3} - \frac{\pi}{12} - 1\right).$$

(b) Using mathematical induction, prove that $1 + e^{i\theta}$ is a factor of $\sum_{r=0}^{2n+1} e^{ir\theta}$ for $n \in \mathbb{Z}^+$, where $1 + e^{i\theta} \neq 0$. **3**

(c) A car of mass M kilograms is travelling at a constant speed of u m/s.
The car is driven onto a sandy beach and experiences a resistance force of Mkv newtons, where k is a positive constant.
The car comes to rest after travelling D metres along the beach.

(i) Show that $k = \dfrac{u}{D}$. **3**

(ii) How many seconds after driving onto the beach was the car travelling at a speed $\dfrac{1}{u}$ m/s ? **3**

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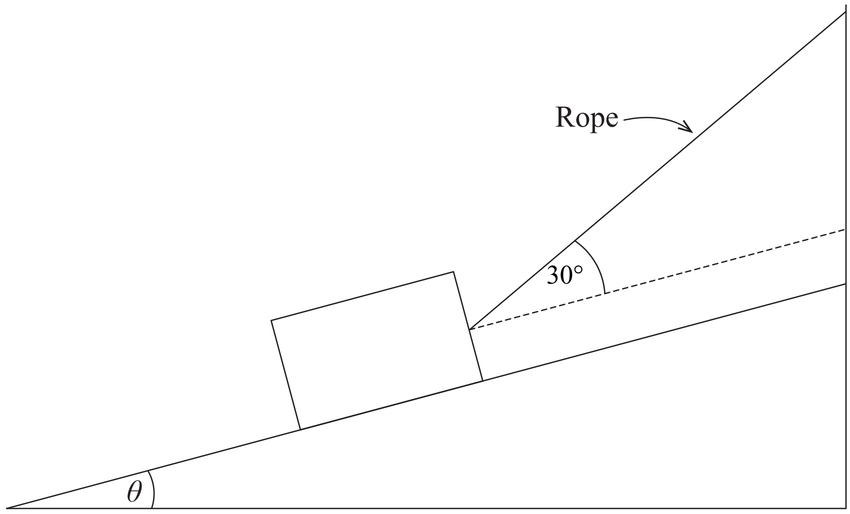
Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Let $f(x)$ be a differentiable function. **2**

Let $G(t) = \int_0^t x f(x)\, dx$, where $t > 0$, such that $G(t^2) = \frac{2}{5}t^5$.

By considering $\frac{d}{dt}G(t^2)$, or otherwise, find $f\left(\frac{4}{25}\right)$.

(b) An object with a mass of 12 kg lies on a frictionless inclined plane.
A rope is attached to the object at an angle of 30° above the plane, as shown.



The tension in the rope, **T** newtons, prevents the object from moving.

(a) Show, by resolving parallel to the plane, that $12g\sin\theta = |\mathbf{T}|\cos 30°$, where g is the acceleration due to gravity. **2**

(b) When the rope is detached, the object moves down the plane with an acceleration of 5.6 m/s^2. **1**
Determine the exact value of the magnitude of **T**.

(c) A sequence is defined by the relationship **4**

$$u_1 = 1,\ u_{n+1} = \frac{1}{2}\left(u_n + \frac{2}{u_n}\right), \text{ where } n \in \mathbb{Z}^+.$$

Use mathematical induction to show that $\dfrac{u_n - \sqrt{2}}{u_n + \sqrt{2}} = \left(\dfrac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}}$.

Question 14 is continued on page 11

(d) A particle of mass 1 kg is projected vertically upward under gravity with speed $2c$ in a medium in which the resistance to motion is $\frac{g}{c^2}$ times the square of the speed, where c is a positive constant.
The acceleration due to gravity is g m/s^2.

(i) Show that the highest point reached, H, is given by **3**

$$H = \frac{c^2}{2g} \ln 5.$$

(ii) Show that the speed, w, with which the particle returns to its starting point is **3**

$$w = \frac{2c}{\sqrt{5}} \text{ m/s}.$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ for integers $n \geq 0$.

(i) Show that, for $n \geq 2$, $nI_n = (n-1)I_{n-2}$ **2**

(ii) Hence determine the exact value of **3**

$$\int_0^{\frac{1}{2}\pi} \cos^6 x \sin^2 x \, dx .$$

(b) A body of unit mass is projected on level ground at an initial speed u m/s inclined at an angle α to the horizontal.
It experiences an air resistance equal to

$$k \, \dot{\underset{\sim}{r}}(t),$$

where $\underset{\sim}{r}(t)$ is the position vector of the mass, $k \in \mathbb{R}^+$, and t is the time in seconds after release. You may assume that the position vector, $\underset{\sim}{r}(t)$, of the body is given by

$$\underset{\sim}{r}(t) = \frac{u \cos \alpha}{k}\left(1 - e^{-kt}\right) \underset{\sim}{i} + \left[\frac{g + ku \sin \alpha}{k^2}\left(1 - e^{-kt}\right) - \frac{g}{k}t\right] \underset{\sim}{j},$$

where g is the acceleration due to gravity.

An identical body is projected simultaneously from the same point in the same direction with velocity U, where $U > u$.
The slower body hits the ground at a point B on the same level as the point of projection O and makes an angle θ with the horizontal.
At that instant the faster body just clears a wall of height h metres above the level of projection.

(i) Show that, while both bodies are in flight, the line joining them makes the same angle α with the horizontal. **2**

(ii) By considering the distance between B and the base of the wall, show that **3**

$$e^{-kT} = \frac{(U - u) \sin \alpha - kh}{(U - u) \sin \alpha},$$

where T is time taken for the slower body to reach B.

(iii) Hence, show that **2**

$$\tan \theta = \tan \alpha + \frac{gh}{u \cos \alpha \left[kh - (U - u) \sin \alpha\right]}$$

Question 15 continues on page 13

(c) A particle is moving in simple harmonic motion with period T seconds and amplitude A cm. By deriving an expression for v^2, where v is the velocity of the particle, or otherwise, show that its maximum velocity is **3**

$$\frac{2\pi A}{T} \text{ cm/s.}$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) Let $P(z) = az^4 + ibz^3 + cz^2 + idz + e$, where a, b, c, d, e, p, and q are real constants. **2**

Let $w \in \mathbb{C}$, such that $P(w) = 0$.

Show that $P(-\overline{w}) = 0$.

(b) Given three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, define

$$\vec{u} = \left(\vec{b} \cdot \vec{c}\right)\vec{a} - \left(\vec{c} \cdot \vec{a}\right)\vec{b}$$
$$\vec{v} = \left(\vec{a} \cdot \vec{c}\right)\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c}$$
$$\vec{w} = \left(\vec{b} \cdot \vec{a}\right)\vec{c} - \left(\vec{c} \cdot \vec{b}\right)\vec{a}$$

The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a triangle.

(i) Prove that $\vec{u}$, $\vec{v}$, and $\vec{w}$ also form a triangle. **2**

(ii) Calculate $\vec{u} \cdot \vec{c}$. **1**

(iii) Prove that the two triangles formed by $\vec{a}$, $\vec{b}$, & $\vec{c}$ and $\vec{u}$, $\vec{v}$, & $\vec{w}$ are similar. **2**

Question 16 continues on page 15

Question 16 (continued)

(c) Consider the integral $I_n = \int_0^1 x^{2n+1} e^{-x^2}\, dx$.

(i) Show that $I_n = -\frac{1}{2e} + nI_{n-1}$, for $n \ge 1$. **2**

(ii) Show that $I_0 = \frac{1}{2} - \frac{1}{2e}$. **1**

(iii) Prove by mathematical induction that for all $n \ge 1$, **3**

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} = e - \frac{2eI_n}{n!}.$$

(iv) Explain why $0 \le I_n \le 1$ for $0 \le x \le 1$. **1**

(v) Deduce that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$. **1**

End of paper

SYDNEY
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2023

YEAR 12

TASK 4 – THSC

Mathematics Extension 2

Sample Solutions

NOTE: Some of you may be disappointed with your mark.
This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done. Writing something down does not justify that your working can be linked to a mark.

Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

MC Answers

1	A	**6**	D
2	B	**7**	A
3	C	**8**	C
4	A	**9**	B
5	A	**10**	D

Section I Multiple Choice Solutions

1 A cubic equation with integer coefficients has $2 - 3i$ and 4 as two of its roots.

What is the third root?

A. $2 + 3i$ B. $3 - 2i$ C. $3 + 2i$ D. $-2 + 3i$

As the coefficients are real then the conjugate is also a root $2 + 3i$

2 Which expression below is equivalent to

$$\frac{e^{-\frac{5i\pi}{6}}}{e^{\frac{i\pi}{2}}}\ ?$$

A. $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

B. $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

C. $-\frac{\sqrt{3}}{2}-\frac{1}{2}i$

D. $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$

$$\frac{e^{-\frac{5i\pi}{6}}}{e^{\frac{i\pi}{2}}}=e^{\frac{-4i\pi}{3}}$$

$$=\cos\left(\frac{-4i\pi}{3}\right)+i\sin\left(\frac{-4i\pi}{3}\right)$$

$$=\cos\left(\frac{4i\pi}{3}\right)-i\sin\left(\frac{4i\pi}{3}\right)$$

$$=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

3 The indefinite integral

$$\int \frac{a-3x}{4-x^2}\,dx$$

can be determined using the partial fraction decomposition

$$\frac{2}{2+x}-\frac{1}{2-x}.$$

What is the value of a ?

A. -2

B. -1

C. 2

D. 3

$$\frac{2}{2+x}-\frac{1}{2-x}=\frac{2(2-x)-(2+x)}{(2+x)(2-x)}$$

$$=\frac{2-3x}{4-x^3}$$

4 A particle is moving along the x-axis, initially moving to the left from the origin.
Its velocity and acceleration are given by

$$v^2 = 2\ln(3+\cos x) \text{ and } a = -\frac{\sin x}{3+\cos x}.$$

Which of the following describes its subsequent motion?

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B. It moves only to the left, alternately slowing to a stop and then speeding up.

C. It slows to a stop, then heads to the right forever.

D. It oscillates between two points

$v \neq 0$ as $2 \leq 3+\cos x \leq 4$

5 A particle of mass m kg is projected vertically upward with a velocity $v\,\mathbf{j}$ m/s.

The magnitude of the air resistance is given by $\dfrac{mgv^2}{\lambda^2}$, where λ is a constant.

Which expression describes the acceleration for the upward motion of the particle?

A. $-\dfrac{g}{\lambda^2}\left(\lambda^2+v^2\right)\mathbf{j}$

B. $-\dfrac{mg}{\lambda^2}\left(\lambda^2+v^2\right)\mathbf{j}$

C. $\dfrac{g}{\lambda^2}\left(\lambda^2-v^2\right)\mathbf{j}$

D. $\dfrac{mg}{\lambda^2}\left(\lambda^2-v^2\right)\mathbf{j}$

$$m\underset{\sim}{a}=-\left(mg+\frac{mgv^2}{\lambda^2}\right)\mathbf{j}$$

$$\therefore \underset{\sim}{a}=-\left(g+\frac{gv^2}{\lambda^2}\right)\mathbf{j}$$

6 The velocity of a body moving in a straight line is given by $v=f(x)$,
where x metres is the distance from origin and v is the velocity in metres per second.
What is the acceleration of the body in m/s^2 ?

A. $f'(x)$

B. $f'(v)$

C. $x\,f'(x)$

D. $f(x)\,f'(x)$

$$a=v\frac{dv}{dx}$$

$$=f(x)f'(x)$$

7 A particle is moving in a straight line such that its velocity, in metres per second, is given by

$$v^2 = 20 - 16x - 4x^2,$$

where x is the displacement, in metres, of the particle from a fixed point, O.

Which of the following statements about the particle is true?

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B. The particle moves in simple harmonic motion, oscillating about the centre $x = -2$ with a period of $\frac{\pi}{2}$ and an amplitude of 3 metres.

C. The particle moves in simple harmonic motion, oscillating about the centre $x = 2$ with a period of π and an amplitude of 3 metres.

D. The particle moves in simple harmonic motion, oscillating about the centre $x = 2$ with a period of $\frac{\pi}{2}$ and an amplitude of 3 metres.

$$v^2 = 20 - 16x - 4x^2$$

$$= 4\left(5 - 4x - x^2\right)$$

$$= 4\left[9 - \left(x+2\right)^2\right]$$

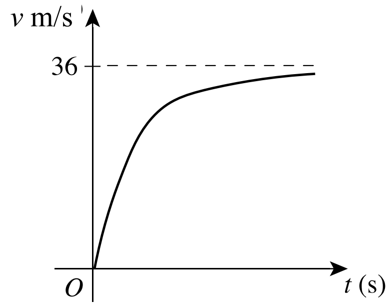
$$n^2 = 4 \Rightarrow T = \frac{2\pi}{n} = \pi$$

8 A mass of 1 kg is dropped from a height in a resistive medium under a constant gravitational acceleration of 10 m/s^2.

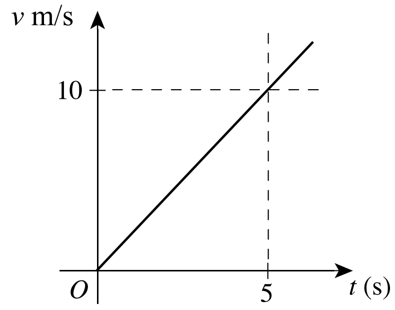
The resistive force is directly proportional to the speed v m/s.

If the constant of proportionality is 0.5, which of the following best represents the velocity-time graph of the mass?

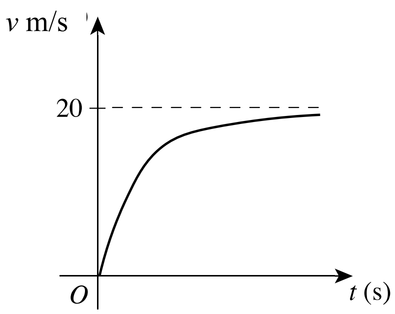
A.



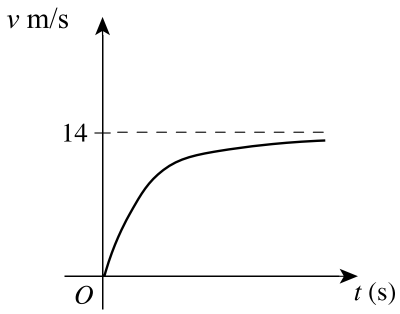
B.



C.



D.



$a = 10 - \frac{1}{2}v$

Terminal velocity when a = 0 i.e. $v_T = 20$

9 What is the value of $\int_a^{a+b} \frac{g(2a+b-x)}{g(2a+b-x)-g(x)}\,dx$?

A. $\frac{a}{2}$

B. $\frac{b}{2}$

C. a

D. b

Let $I = \int_a^{a+b} \frac{g(x)}{g(2a+b-x)-g(x)}dx$

$$\int_a^{a+b} \frac{g(2a+b-x)}{g(2a+b-x)-g(x)}\,dx = \int_a^{a+b} \frac{g(2a+b-x)-g(x)+g(x)}{g(2a+b-x)-g(x)}\,dx$$

$$= \int_a^{a+b} 1+\frac{g(x)}{g(2a+b-x)-g(x)}dx$$

$$= (a+b-a)+\int_a^{a+b} \frac{g(x)}{g(2a+b-x)-g(x)}dx$$

$$= b+\int_a^{a+b} \frac{g(x)}{g(2a+b-x)-g(x)}dx$$

Let $u = 2a+b-x \Rightarrow du = -dx$

$x : a \sim a+b$

$u : a+b \sim a$

$$\int_a^{a+b} \frac{g(x)}{g(2a+b-x)-g(x)}dx = \int_{a+b}^{a} \frac{g(2a+b-u)}{g(u)-g(2a+b-u)}(-du)$$

$$= \int_a^{a+b} \frac{g(2a+b-u)}{g(u)-g(2a+b-u)}du$$

$$= -I$$

$\therefore I = b-I \Rightarrow I = \frac{b}{2}$

10 Given that ϕ is a complex number such that Re (ϕ) > Im (ϕ) > 1.

Which of the following can be true?

A. $|\phi| = \sqrt{2\,\text{Re}(\phi)\,\text{Im}(\phi)}$

B. $\left|\dfrac{\phi - \overline{\phi}}{\phi + \overline{\phi}}\right| > 1$

C. $\left|\text{Im}\left(\dfrac{\phi}{i}\right)\right| < \text{Im}(\phi)$

D. $|\phi| < \text{Re}(\phi) + \text{Im}(\phi)$

Do a numerical value e.g. $\phi = 3 + 2i$

OR

Let x = Re (ϕ) and y = Im(ϕ)

A. FALSE

$$|\phi| = \sqrt{x^2 + y^2} = \sqrt{2xy} \text{ if } x^2 + y^2 = 2x \Rightarrow (x - y)^2 = 0$$
$$\therefore x = y$$

B. FALSE

$$\left|\frac{\phi - \overline{\phi}}{\phi + \overline{\phi}}\right| = \left|\frac{2iy}{2ix}\right| = \left|\frac{y}{x}\right| = \frac{y}{x}$$
$$\left|\frac{\phi - \overline{\phi}}{\phi + \overline{\phi}}\right| > 1 \Rightarrow \frac{y}{x} > 1$$
$$\therefore y > x$$

C. FALSE

$$\phi = x + iy \Rightarrow -i\phi = y - ix$$
$$\left|\text{Im}\left(\frac{\phi}{i}\right)\right| = |\text{Im}(-i\phi)| = |-\text{Re}(\phi)| = |\text{Re}(\phi)|$$
$$\left|\text{Im}\left(\frac{\phi}{i}\right)\right| < \text{Im}(\phi) \Leftrightarrow x < y$$

D. TRUE

$$|\phi| = \left|\sqrt{x^2 + y^2}\right| = \left|\sqrt{(x+y)^2 - 2xy}\right| < \left|\sqrt{(x+y)^2}\right| = x + y$$

Q11) a) $\frac{1}{u} + \frac{1}{v} = \frac{1}{1+2i} + \frac{1}{2+i}$

$= \frac{1}{1+2i} \times \frac{1-2i}{1-2i} + \frac{1}{2+i} \times \frac{2-i}{2-i}$

$= \frac{1-2i}{1^2+2^2} + \frac{2-i}{2^2+1^2}$

$= \frac{3-3i}{5}$

$= \frac{3(1-i)}{5} \times \frac{1+i}{1+i}$

$= \frac{3(1^2+1^2)}{5+5i}$

$= \frac{6}{5+5i} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{6\sqrt{2}}{5\sqrt{2}+5\sqrt{2}i}$

$\therefore w = 5\sqrt{2} + 5\sqrt{2}i$

COMMENT: This was generally done well

b)i) $z = ke^{i\alpha}$

$z^3 = k^3 e^{i3\alpha}$

$w = 3k^2 e^{i4\alpha}$

$\bar{w} = 3k^2 e^{-i4\alpha}$

$\frac{z^3}{\bar{w}} = \frac{k^3 e^{i3\alpha}}{3k^2 e^{-i4\alpha}}$

$= \frac{k}{3} e^{i3\alpha-(-i4\alpha)}$

$= \frac{k}{3} e^{i7\alpha}$ Note: $\frac{k}{3} > 0$ and $-\pi < 7\alpha \leq \pi$.

ii) $\operatorname{Im}\left(\left(\frac{z^3}{\overline{\omega}}\right)^n\right) = 0$

let $\alpha = \frac{\pi}{21}$

$$\operatorname{Im}\left(\left(\frac{k}{3}e^{i7\left(\frac{\pi}{21}\right)}\right)^n\right) = 0$$

$$\operatorname{Im}\left(\left(\frac{k}{3}\right)^n e^{\frac{i\pi n}{3}}\right) = 0$$

$$\operatorname{Im}\left(\left(\frac{k}{3}\right)^n \left(\cos\left(\frac{\pi n}{3}\right) + i\sin\left(\frac{\pi n}{3}\right)\right)\right) = 0$$

$$\left(\frac{k}{3}\right)^n \sin\left(\frac{\pi n}{3}\right) = 0$$

$$\frac{\pi n}{3} = m\pi \quad \text{where } m \in \mathbb{Z}$$

$$n = 3m$$

n must be a multiple of 3.

COMMENT: When introducing a pronumeral it should be defined (ie $m \in \mathbb{Z}$). It would have been nice for more students to interpret their answer (ie. n is a multiple of 3)

c) i) $\frac{d}{dx}\left(e^{\cos 2x}\right) = -2\sin 2x\, e^{\cos 2x}$

ii) $$\int e^{\cos 2x} \sin 4x\, dx = \int e^{\cos 2x}(2\sin 2x \cos 2x)\, dx$$

$$= -\int \cos 2x . \left(-2\sin 2x\, e^{\cos 2x}\right) dx$$

$$= -\left[\cos 2x . e^{\cos 2x} - \int e^{\cos 2x} . (-2\sin 2x)\, dx\right]$$

$$= -\cos 2x . e^{\cos 2x} + \int (-2\sin 2x) e^{\cos 2x}\, dx$$

$$= -\cos 2x . e^{\cos 2x} + e^{\cos 2x} + C$$

$$= e^{\cos 2x}\left(1 - \cos 2x\right) + C$$

iii) $\int e^{\cos 2x}(\sin x \cos 3x)\,dx$

$$= \int e^{\cos 2x}(\cos 3x \sin x)\,dx$$

$$= \int e^{\cos 2x} \cdot \frac{1}{2}\left(\sin(3x+x) - \sin(3x-x)\right)dx$$

$$= \frac{1}{2}\int e^{\cos 2x}(\sin 4x - \sin 2x)\,dx$$

$$= \frac{1}{2}\int e^{\cos 2x}\sin 4x\,dx - \frac{1}{2}\int \sin 2x . e^{\cos 2x}\,dx$$

$$= \frac{1}{2}\int e^{\cos 2x}\sin 4x\,dx + \frac{1}{4}\int -2\sin 2x\, e^{\cos 2x}\,dx$$

$$= \frac{1}{2}e^{\cos 2x}\left(1 - \cos 2x\right) + \frac{1}{4}e^{\cos 2x} + C$$

OR $\frac{3}{4}e^{\cos 2x} - \frac{1}{2}\cos 2x\, e^{\cos 2x} + C$

COMMENT: Given the structure of the question students should be looking to use the previous part(s). Given the number of marks allocated to part (ii) & (iii) it won't necessarily be immediately obvious how to use the previous part(s).

Ext 2 2023 THSC: Notes and Solutions for Question 12

Notes on part a) : The best way to do this problem is the first method shown. Otherwise, a lot of scope for arithmetic errors to creep in was made available - and creep in they did . Only one pupil (from memory) chose the "first" method.
Some pupils chose, for their substitution, a non-injective function (many-to-one) over the interval of the definite-integral and their new boundary became [0, 0]. Make sure your substitution is injective (one-to-one) over its restricted Domain (the original boundary).

Notes on part b) : Realising that the derivative of the natural log of x is already one of the products in the function, helped most pupils . Those that didn't see this struggled to find a solution and some couldn't at all.
Even so, some were not able to find the correct values of the numerators of the partial fractions even after setting it out more thoroughly than shown in the solutions. They made arithmetic errors on simple linear equations.

Notes on part c) : For part i) students needed to plot the real part of both numbers on Re = 1 and "show" that the imaginary part of one was greater than 1 and between 0 and -1 for the other. Too many pupils did not "show" this and left it as assumed. This was a mistake, it should not be left to be “assumed".

For part ii) the numbers had to be the same distance from the origin as Z_1, and collinear with the origin and Z_2.

For part iii) beta is defined as an angle in the clockwise direction, whereas alpha is in an anticlockwise direction. k = beta - alpha was the correct answer for one of the values of k and the other was a rotation by pi of the same value. Those that had plus or minus (beta + alpha) were wrong! However, it occurred so often that I let it go (marked it wrong but begrudgingly gave the mark). If your script says “see sols” in this section you were given a reprieve. Put it this way, if John Glen was relying on YOU, he would NOT have made it. Those who got it right may ‘flex’ to their classmates for this occasion only.

For part iv) pupils needed to state the answer in the terms the question told them to. Too many pupils used the magnitude of the "vectors" between the numbers but this was not helpful.

(a) Evaluate **3**

$$\int_1^9 (x-1)\sqrt{16-(x-5)^2}\ dx.$$

Let $u = x - 5$ and $du = dx.$

When $x = 9, u = 4.$
When $x = 1, u = -4.$

Integral becomes: $\displaystyle\int_{-4}^{4} (u+4)\sqrt{16-u^2}du$

$$= 4\int_{-4}^{4} \sqrt{16-u^2}du + \int_{-4}^{4} u\sqrt{16-u^2}du$$

$$= 4 \times \frac{\pi 4^2}{2} + 0 \quad *$$

$$= 32\pi$$

$*$ The justification for this is that this is the sum of four times the area of a semi-circle and an odd function on an interval whose centre is the origin.

Method 2: Let $x - 5 = 4\sin\theta,$ then $dx = 4\cos\theta d\theta$ and $x - 1 = 4(1+\sin\theta).$

When $x = 9,\ \theta = \frac{\pi}{2}.$ When $x = 1,\ \theta = -\frac{\pi}{2}.$

$$I = 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)\sqrt{16-16\sin^2\theta} \times \cos\theta\, d\theta$$

$$= 64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)\cos^2\theta\, d\theta$$

$$= 64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2\theta - \sin\theta\cos^2\theta\right) d\theta$$

$$= \frac{64}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1+\cos 2\theta + 2(-\sin\theta)\cos^2\theta\right)\ d\theta$$

$$= 32\left(\theta + \frac{\sin 2\theta}{2} + \frac{2}{3}\cos^3\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 32\left(\frac{\pi}{2} - -\frac{\pi}{2} + 0 - 0 + \frac{2}{3}(0-0)\right)$$

$$= 32\pi$$

(b) Find $\int \frac{1+\ln x}{x(2+\ln x)(3+\ln x)}\,dx$ **4**

$$= \int \frac{1}{x}\left(\frac{A}{2+\ln x} + \frac{B}{3+\ln x}\right)dx \qquad B = 2,\ A = -1 \text{ by inspection.}$$

$$= \int \frac{1}{x}\left(\frac{2}{3+\ln x} - \frac{1}{2+\ln x}\right)dx \qquad \text{Let } u = \ln x,\ \frac{du}{dx} = \frac{1}{x}$$

$$= \int \left(\frac{2}{3+u} - \frac{1}{2+u}\right)\frac{du}{dx}dx$$

$$= 2\ln(3+u) - \ln(2+u) + C$$

$$= 2\ln(3+\ln x) - \ln(2+\ln x) + C$$

Method 2: Let $u = \ln x,\ \frac{du}{dx} = \frac{1}{x}$

$$I = \int \frac{1+u}{(2+u)(3+u)}du$$

$$= \int \frac{1+u}{6+5u+u^2}du$$

$$= \frac{1}{2}\int \left(\frac{5+2u}{u^2+5u+6} - \frac{3}{u^2+5u+6}\right)du$$

$$= \frac{1}{2}(\ln(u+3) + \ln(u+2)) - \frac{3}{2}\int \frac{1}{(u+2)(u+3)}du$$

$$= \frac{1}{2}(\ln(3+\ln x) + \ln(2+\ln x)) - \frac{3}{2}\int \left(\frac{1}{u+2} - \frac{1}{u+3}\right)du\ *$$

$$= \frac{1}{2}(\ln(3+\ln x) + \ln(2+\ln x)) + \frac{3}{2}(\ln(3+u) - \ln(2+u)) + C$$

$$= 2\ln(3+\ln x) - \ln(2+\ln x) + C$$

$$*\ \frac{1}{u+2} + \frac{-1}{u+3} = \frac{1}{(u+2)(u+3)} \qquad \text{by inspection.}$$

Method 3: Shift the function and the boundaries to the left by five.

$$\begin{aligned}
I &= \int_{-4}^{4} (x+4)\sqrt{16-x^2}\,dx \\
&= \int_{-4}^{4} \left(x\sqrt{16-x^2}+4\sqrt{16-x^2}\right) dx \\
&= \int_{-4}^{4} (f(x)+g(x))\,dx, \quad \text{where} \quad \begin{cases} f(x) = x\sqrt{16-x^2} \\ g(x) = 4\sqrt{16-x^2} \end{cases}
\end{aligned}$$

$$\int_{-4}^{4} f(x)\,dx \Rightarrow -\frac{1}{3}\int_{-4}^{4} -2x \times \frac{3}{2}\sqrt{16-x^2}dx = -\frac{1}{3}\left(16-x^2\right)^{\frac{3}{2}}\Big|_{-4}^{4} = 0$$

$$\int_{-4}^{4} g(x)\,dx \Rightarrow$$ Either four times the area of a semi-circle of radius four or use a trigonometric substitution (both of which were used in earlier methods).

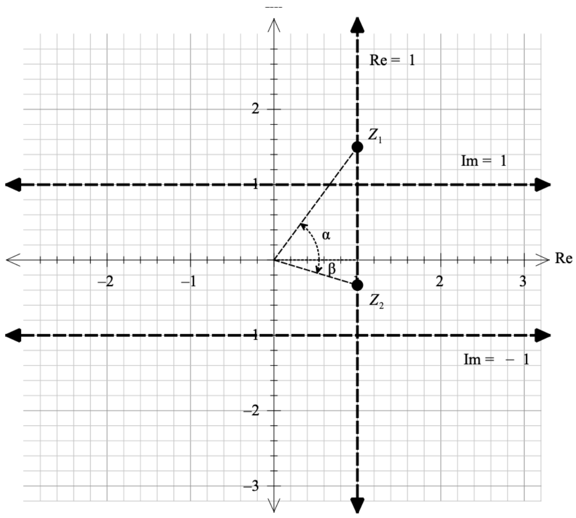
(c) The complex numbers z_1 and z_2 are such that

$$z_1 = 1 + bi,\ b > 1,\ \operatorname{Arg}(z_1) = \alpha \text{ and}$$

$$z_2 = 1 - ci,\ 0 < c < 1,\ \operatorname{Arg}(z_2) = \beta$$

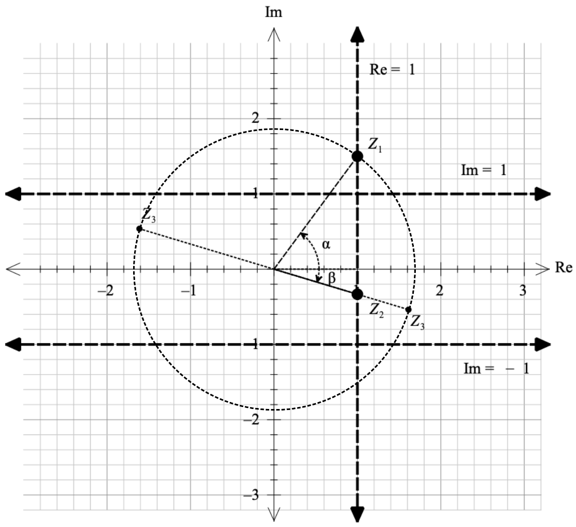
Let Z_1 and Z_2 be points representing z_1 and z_2 respectively on the Argand diagram.

(i) Indicate Z_1 and Z_2 on an Argand diagram. **2**



Z_3 is the point representing z_3 on an Argand diagram such that $|z_3| = |z_1|$, the origin O is collinear with Z_2 and Z_3, and $z_3 = e^{ki} z_1$, where k is a real constant.

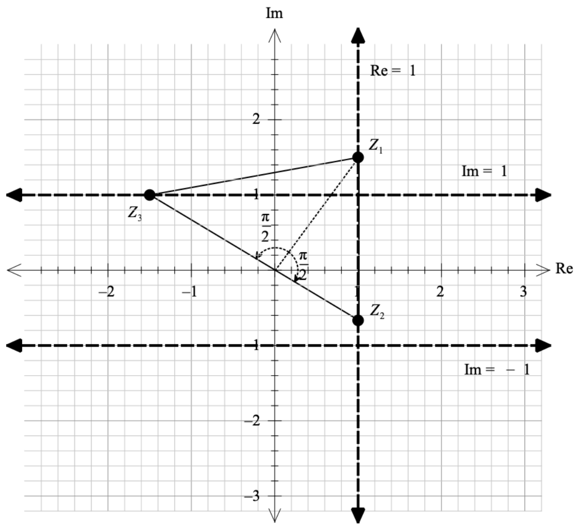
(ii) Indicate the two possible positions of Z_3 on the same Argand diagram drawn in part (i). **2**



(iii) Hence determine the two possible values of k, leaving your answers in terms of α and β. **2**

$$e^{ik} = e^{i(\beta-\alpha)}$$
$$\text{or} \quad e^{ik} = e^{i(\beta-\alpha+\pi)}$$
$$k = \beta - \alpha, \beta - \alpha + \pi$$

(iv) For the value of $k = \dfrac{\pi}{2}$, express the area of triangle $Z_1Z_2Z_3$, in terms of $|z_1|$ and $|z_2|$. **2**



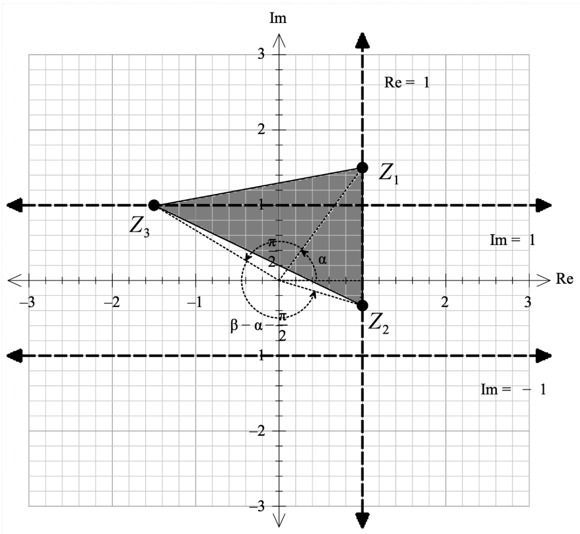
$$Z_3 = e^{i\frac{\pi}{2}} Z_1 = iZ_1, \text{ which is a rotation about the origin by } \frac{\pi}{2}.$$

Since $0, Z_2,$ and Z_3 are collinear, $\angle Z_3OZ_1$ and $\angle Z_10Z_2$ sum to π and $\angle Z_10Z_2 = \dfrac{\pi}{2}$.

$$\text{Area of } \triangle Z_1Z_2Z_3 = \frac{|z_1|}{2}\left(|z_1| + |z_2|\right)$$

For interest's sake only, here is what happens when the argument between Z_1 and Z_3 is a right angle but Z_2 is not collinear with the origin and Z_3. Of course the magnitude of Z_1 and Z_3 are still equal.

(iv) For the value of $k = \frac{\pi}{2}$, express the area of triangle $Z_1Z_2Z_3$, in terms of $|z_1|$ and $|z_2|$. **2**



$$\text{Area of } \triangle OZ_1Z_3 = \frac{|z_1|^2}{2}$$
$$\text{Area of } \triangle OZ_2Z_1 = \frac{1}{2}\sin(\alpha-\beta)\,|z_1|\,|z_2|$$
$$\text{Area of } \triangle OZ_3Z_2 = \frac{1}{2}\sin\left(\beta-\alpha-\frac{\pi}{2}\right)|z_1|\,|z_2|$$

$$\text{Area of } \triangle Z_1Z_2Z_3 = \frac{|z_1|}{2}\left(|z_1| + |z_2|\left(\sin(\alpha-\beta)+\sin\left(\beta-\alpha-\frac{\pi}{2}\right)\right)\right)$$

Consider the line interval made by the points Z_1 and Z_2.

When this line is above the origin, as is the case in the image, the angle is greater than 180 degrees and the area is negative. When the line is below the origin the angle is less than 180 degrees and the area is positive.

This area may be rewritten as $\frac{|z_1|}{2}\left(|z_1| + |z_2|\left(\sin(\alpha-\beta)+\cos(\alpha-\beta)\right)\right)$,

or $\frac{|z_1|}{2}\left(|z_1| + |z_2|\sqrt{2}\sin\left(\alpha-\beta+\frac{\pi}{4}\right)\right)$

Note: When $\alpha-\beta = \frac{\pi}{2}, \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$.

Question 13 (16 marks) SOLUTIONS

(a) A particle is initially at $x = 1$ with a velocity of 2 m/s.
The acceleration of the particle is given by

$$a = \frac{1}{2}\left(1 - \frac{1}{x^2}\right) \text{ m/s}^2,$$

where x is the displacement of the particle from O.

(i) Prove that $\dfrac{dx}{dt} = \dfrac{1+x}{\sqrt{x}}$. **3**

$$a = v\frac{dv}{dx} = \frac{1}{2}\left(1 - x^{-2}\right)$$

$$\therefore 2v\,dv = \left(1 - x^{-2}\right)dx$$

$$\therefore \int_2^v 2V\,dV = \int_1^x 1 - X^{-2}\,dX$$

$$\therefore \left[V^2\right]_2^v = \left[X + \frac{1}{X}\right]_1^x$$

$$\therefore v^2 - 4 = \left(x + \frac{1}{x}\right) - (1+1)$$

$$\therefore v^2 = \frac{x^2 + 2x + 1}{x}$$

$$= \frac{(x+1)^2}{x}$$

$$\therefore v = \pm\frac{x+1}{\sqrt{x}}$$

As $v = 0$ only at $x = -1$, and initially it is travelling to the right then $v = \dfrac{x+1}{\sqrt{x}}$.

Comments

Students who could not justify why the velocity was positive could not get full marks.
Most students tried to justify by using initial data, i.e. $v > 0$ at $t > 0$, but this is wrong.
The particle can only get a negative velocity if it stops.

For this question some students tried integrating wrt t, but then they got nonsense, which was marked as such.

(a)	(ii)	Show that the time, in seconds, taken for the particle to reach $x = 3$ is $$2\left(\sqrt{3} - \frac{\pi}{12} - 1\right).$$	**4**

Let T be the time needed.

$$\frac{dx}{dt} = \frac{x+1}{\sqrt{x}}$$

$$\therefore \frac{\sqrt{x}}{x+1}\,dx = dt$$

$$\therefore \int_1^3 \frac{\sqrt{x}}{x+1}\,dx = \int_0^T dt$$

Let $x = u^2 \Rightarrow dx = 2u\,du$

x: $1 \sim 3$

u: $1 \sim \sqrt{3}$

$$\therefore T = \int_1^{\sqrt{3}} \frac{u}{u^2+1}\,2u\,du$$

$$= 2\int_1^{\sqrt{3}} \frac{u^2}{u^2+1}\,du = 2\int_1^{\sqrt{3}} \frac{u^2+1-1}{u^2+1}\,du$$

$$= 2\left[\int_1^{\sqrt{3}} 1\,du - \int_1^{\sqrt{3}} \frac{1}{u^2+1}\,du\right]$$

$$= 2\left[\sqrt{3} - 1 - \left(\tan^{-1}\sqrt{3} - \tan^{-1}1\right)\right]$$

$$= 2\left(\sqrt{3} - 1 - \frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$=2\left(\sqrt{3} - 1 - \frac{\pi}{12}\right)$$

Comments

This was generally well done, though some students made it harder by doing a trigonmetric substitition. Some students had a problem with the upper and lower bounds. When you make a substitution the bounds need to change.

(b) Using mathematical induction, prove that $1+e^{i\theta}$ is a factor of $\sum_{r=0}^{2n+1} e^{ir\theta}$ for $n \in \mathbb{Z}^+$, where $1+e^{i\theta} \neq 0$. **3**

Test $n = 1$:

$$\sum_{r=0}^{3} e^{ir\theta} = 1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}$$

$$= 1+e^{i\theta}+e^{2i\theta}(1+e^{i\theta})$$

$$= (1+e^{i\theta})\left(1+e^{2i\theta}\right)$$

So it is true for $n = 1$.

Assume true for $n = k$ i.e. $1+e^{i\theta}$ is a factor of $\sum_{r=0}^{2k+1} e^{ir\theta}$.

Need to prove true for $n = k+1$ i.e. show $1+e^{i\theta}$ is a factor of $\sum_{r=0}^{2k+3} e^{ir\theta}$.

$$\sum_{r=0}^{2k+3} e^{ir\theta} = \sum_{r=0}^{2k+1} e^{ir\theta} + e^{(2k+2)i\theta} + e^{(2k+3)i\theta}$$

$$= \sum_{r=0}^{2k+1} e^{ir\theta} + e^{(2k+2)i\theta}(1+e^{i\theta})$$

By assumption $1+e^{i\theta}$ is a factor of $\sum_{r=0}^{2k+1} e^{ir\theta}$, so $1+e^{i\theta}$ is factor of $\sum_{r=0}^{2k+3} e^{ir\theta}$.

So by the principle of mathematical induction, the statement is true for $n \in \mathbb{Z}^+$.

Comments

Students who did a divisibility question the same way as in ME 1 and wrote something like $M\left(1+e^{i\theta}\right)$, where $M \in \mathbb{Z}$ (or anything except $\mathbb{C}$) could not get full marks.

A lot of students have forgotten their Stage 4 factorising knowledge.

(c) A car of mass M kilograms is travelling at a constant speed of u m/s.
The car is driven onto a sandy beach and experiences a resistance force of Mkv newtons, where k is a positive constant.
The car comes to rest after travelling D metres along the beach.

(i) Show that $k = \frac{u}{D}$. **3**

Let $t = 0$ and $x = 0$ when the car starts to drive on to the beach i.e. $t = 0$, $x = 0$, $v = u$

The car travelling at a constant speed initially means that the net force on the car before driving onto the beach is 0 newtons.

$$\therefore M\ddot{x} = -Mkv$$

$$\therefore \ddot{x} = -kv$$

$$\therefore v\frac{dv}{dx} = -kv \Rightarrow \frac{dv}{dx} = -k$$

$$\therefore v = -kx + C \Rightarrow v = -kx + u \quad [x = 0,\ v = u]$$

Now, $v = 0$ when $x = D$

$$\therefore 0 = -kD + u \Rightarrow k = \frac{u}{D}$$

Comments

This was very well done – except for force diagrams.

(c) (ii) How many seconds after driving onto the beach was the car travelling at a speed $\frac{1}{u}$ m/s ? **3**

Let the time be T seconds.

$$\ddot{x} = -kv$$

$$\therefore \frac{dv}{dt} = -\frac{u}{D}v$$

$$\therefore \frac{dv}{v} = -\frac{u}{D}dt$$

$$\therefore \int_u^{\frac{1}{u}} \frac{dv}{v} = -\frac{u}{D}\int_0^T dt$$

$$\therefore \ln\frac{1}{u} - \ln u = -\frac{u}{D}(T-0)$$

$$\therefore -\ln u - \ln u = -\frac{u}{D}T$$

$$\therefore T = \frac{2D\ln u}{u}$$

Comments

Students who tried to continue with the answer from (b), rather than start again, had a harder time trying to prove this result. Otherwise, this was very well done.

14. (a) $\frac{d}{dt} G(t^2) = 2t \times G'(t^2)$ (Chain Rule)

$G(t^2) = \int_0^{t^2} x f(x)\, dx$

$\frac{d G(t^2)}{dt} = \frac{d}{dt}\int_0^{t^2} x f(x)\, dx$

$= 2t \times t^2 \times f(t)^2$ (Fundamental theorem of Calculus)

$= 2t^3 f(t^2)$ ①

Also $G(t^2) = \frac{2}{5} t^5$

$\frac{d}{dt}(G(t^2)) = 2t^4$ ②

Equate ① and ②

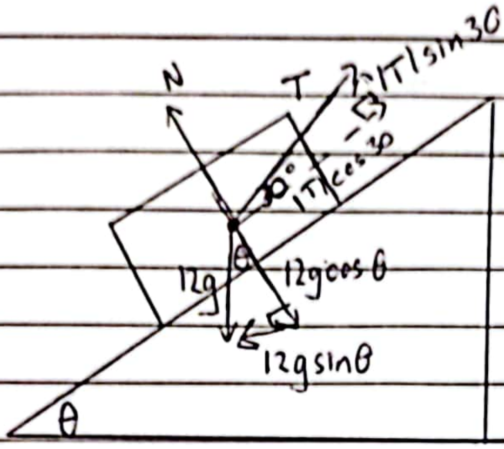
$2t^3 f(t^2) = 2t^4$

$f(t^2) = t \quad (t > 0)$

$\therefore f\left(\frac{4}{25}\right) = \frac{2}{5}$

Marking Scale:	Marker's comments:
2 marks: Correct Solution 1 mark: Progress towards correct answer	* Majority of candidates did not do well in this question as they missed the chain Rule.

14. b) a)



From diagram above, parallel to the plane

$m\ddot{x} = |T|\cos 30 - 12g\sin\theta$

Since object is not moving, $\ddot{x} = 0$

$\therefore 0 = |T|\cos 30 - 12g\sin\theta$

$\therefore 12g\sin\theta = |T|\cos 30$

Marking Scale	Marker's comments.
2 marks : Correct solution by showing the resolution of G component, T component and putting it altogether.	* This was poorly done by many candidates as they did not show any resolution and just restated the question.
1 mark : Some progress in showing the resolution of the components.	

14. b) (b) Rope detached : $F = m\ddot{x}$

$= 12 \times 5.6$

$= 67.2$

$= 12g\sin\theta$

Sub $12g\sin\theta = 67.2$ into (a)

$|T|\cos 30 = 12g\sin\theta$

$|T| \times \frac{\sqrt{3}}{2} = 67.2$

$|T| = \frac{67.2 \times 2}{\sqrt{3}} = \frac{224\sqrt{3}}{5}$ Newtons

Marking Scale	Marker's comments
1 mark : Correct answer	* Majority of candidates did well in this question.
	* Some candidates made arithmetic errors which needs to be addressed.

14.(i) Test for $n=1$

$$LHS: \frac{U_1 - \sqrt{2}}{U_1 + \sqrt{2}} = \frac{1-\sqrt{2}}{1+\sqrt{2}}$$

$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{1}$$

$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^0}$$

$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{1-1}}$$

$$= RHS$$

$\therefore$ True for $n=1$

Assume the statement is true for $n=k \in \mathbb{Z}^+$

i.e. $$\frac{U_k - \sqrt{2}}{U_k + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}$$

To prove true for $n=k+1$

RTP: $$\frac{U_{k+1} - \sqrt{2}}{U_{k+1} + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^k}$$

$$LHS = \frac{U_{k+1} - \sqrt{2}}{U_{k+1} + \sqrt{2}}$$

$$= \frac{\frac{1}{2}\left(U_k + \frac{2}{U_k}\right) - \sqrt{2}}{\frac{1}{2}\left(U_k + \frac{2}{U_k}\right) + \sqrt{2}}$$ (From given information)

$$= \frac{U_k + \frac{2}{U_k} - 2\sqrt{2}}{U_k + \frac{2}{U_k} + 2\sqrt{2}}$$

$$= \frac{U_k^2 - 2\sqrt{2}U_k + 2}{U_k^2 + 2\sqrt{2}U_k + 2}$$

$$= \frac{(U_k - \sqrt{2})^2}{(U_k + \sqrt{2})^2}$$

$$= \left(\frac{U_k - \sqrt{2}}{U_k + \sqrt{2}}\right)^2$$

$$= \left(\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}\right)^2$$

$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1} \times 2^1}$$

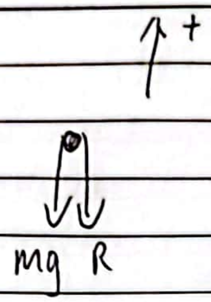
$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^k}$$

$= RHS$.

$\therefore$ The statement is proven true by Mathematical Induction for $n \in \mathbb{Z}^+$

Marking Scale	Marker's Comments.
4 marks: Valid proof by Mathematical Induction.	* Most common error by candidates is not reading the question carefully, particularly the power ie. $\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}$ as $\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2k-1}$
3 marks: Significant progress towards valid proof	
2 marks: Some progress towards valid proof.	* Candidates need to show all steps in proofs. Do not assume that Markers will "guess" how some line of working appear out of nowhere.
1 mark: Verification of case $n=1$	

d) i)



$x = H, \; v = 0$

$x = 0, \; v = u = 2c$

$$m\ddot{x} = -mg - \frac{gv^2}{c^2}$$

Since $m = 1$

$$\ddot{x} = -g - \frac{gv^2}{c^2}$$

$$v\frac{dv}{dx} = -g\left(1 + \frac{v^2}{c^2}\right)$$

$$\frac{dv}{dx} = -g\left(\frac{c^2+v^2}{c^2 v}\right)$$

$$\frac{dx}{dv} = \frac{-1}{g}\left(\frac{c^2 v}{c^2+v^2}\right)$$

$$\int_0^H dx = -\frac{c^2}{2g}\int_{2c}^{0} \frac{2v}{c^2+v^2}\,dv \qquad \frac{2v}{c^2+v^2} > 0$$

$$[x]_0^H = \frac{-c^2}{2g}\left[\ln(c^2+v^2)\right]_{2c}^{0}$$

$$H - 0 = \frac{-c^2}{2g}\left(\ln c^2 - \ln(c^2 + (2c)^2)\right)$$

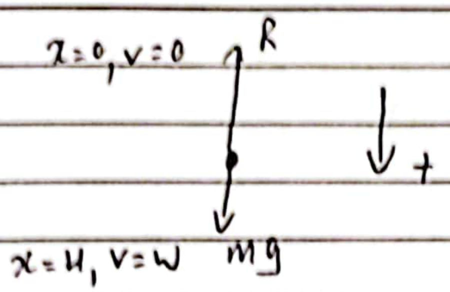
$$H = -\frac{c^2}{2g} \ln c^2 - \ln 5c^2$$

$$= -\frac{c^2}{2g} \ln \frac{c^2}{5c^2}$$

$$= \frac{c^2}{2g} \ln\left(\frac{5c^2}{c^2}\right) = \frac{c^2}{2g} \ln 5$$

Marking Scale	Marker's comments
3 marks: Correct Solution	* Significant number of candidates fudged their answer to get the result. Candidates were penalised for it.
2 marks: Significant progress towards correct answer	
1 mark: Finding correct $\frac{dx}{dv}$	

14. d) (ii)



$$m\ddot{x} = mg - \frac{g}{c^2}v^2$$

As $m = 1$

$$\ddot{x} = g - \frac{g}{c^2}v^2$$

$$\frac{v\,dv}{dx} = \frac{gc^2}{c^2} - \frac{gv^2}{c^2}$$

$$= \frac{g(c^2 - v^2)}{c^2} \qquad (>0)$$

$$\frac{dx}{dv} = \frac{vc^2}{gc^2 - gv^2}$$

$$\int_0^H dx = -\frac{c^2}{2g}\int_0^W \frac{-2v}{c^2-v^2}\,dv$$

$$[x]_0^H = \frac{-c^2}{2g}\left[\ln(c^2-v^2)\right]_0^W$$

$$H - 0 = \frac{-c^2}{2g}(c^2 - w^2) + \frac{c^2}{2g}\ln(c^2)$$

$$H = \frac{c^2}{2g}\ln\left(\frac{c^2}{c^2-w^2}\right)$$

From (i)

$$H = \frac{c^2}{2g}\ln 5$$

$$\frac{c^2}{2g}\ln 5 = \frac{c^2}{2g}\ln\left(\frac{c^2}{c^2-w^2}\right)$$

$$5 = \frac{c^2}{c^2-w^2}$$

$$5c^2 - 5w^2 = c^2$$

$$4c^2 = 5w^2$$

$$w^2 = \frac{4c^2}{5} \quad \text{Since speed} > 0$$

$$\therefore \quad w = \sqrt{\frac{4c^2}{5}}$$

$$= \frac{2c}{\sqrt{5}} \text{ m/s.}$$

Marking Scale	Marker's Comments
- 3 marks: Correct Solution - 2 marks: Significant progress towards correct solution. - 1 mark: Finding correct $\frac{dx}{dV}$	* Some candidates fudged their working to achieve the result; One common mistake was using the wrong boundaries but somehow achieved the correct result. Candidates need to take more care.

Question 15 Solutions

(a) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

(i) $I_n = \int_0^{\frac{\pi}{2}} \cos^{n-1} x . \frac{d}{dx} \sin x \ dx$

$$= [\cos^{n-1} x . \sin x]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x . \sin^2 x \, dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x . (1 - \cos^2 x) dx$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore (n - 1 + 1) I_n = (n-1) I_{n-2}$$

$$n I_n = (n-1) I_{n-2}$$

[This part was well answered by most candidates, though some took a circuitous route through the parts section.]

(ii) $\int_0^{\frac{\pi}{2}} \cos^6 x . \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos^6 x . (1 - \cos^2 x) dx$

$$= I_6 - I_8$$

$$= I_6 - \tfrac{7}{8} I_6 \qquad \text{(From above)}$$

$$= \tfrac{1}{8} \left(\tfrac{5}{6} I_4 \right)$$

$$= \frac{1}{8} \left(\frac{5}{6} \right) \left(\frac{3}{4} I_2 \right)$$

$$= \frac{15}{192} \left(\frac{1}{2} I_0 \right) \qquad I_0 = \frac{\pi}{2}$$

$$= \frac{15\pi}{768} = \frac{5\pi}{256}$$

[Generally very well answered.
Marks were lost for not using the recurrence relation, or for not giving the result in simplest form.]

(b) (i) After t seconds, let the two bodies be at positions P and Q.

$P\left(\frac{u\cos\alpha}{k}(1-e^{-kt}),\ \frac{g+ku\sin\alpha}{k^2}(1-e^{-kt})-\frac{g}{k}t\right)$ (slower body)

$Q\left(\frac{U\cos\alpha}{k}(1-e^{-kt}),\ \frac{g+kU\sin\alpha}{k^2}(1-e^{-kt})-\frac{g}{k}t\right)$ (faster body)

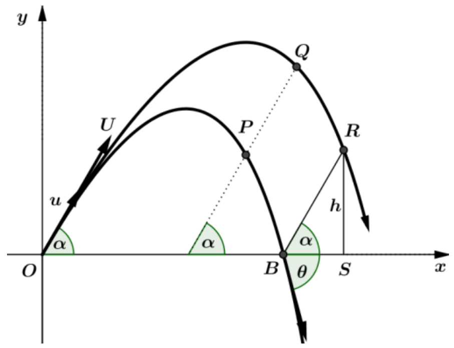
$$\text{Gradient PQ} = \frac{\left(\frac{g+kU\sin\alpha}{k^2}-\frac{g+ku\sin\alpha}{k^2}\right)(1-e^{-kt})}{\left(\frac{U\cos\alpha}{k}-\frac{u\cos\alpha}{k}\right)(1-e^{-kt})}$$

$$= \tan\alpha.$$

Hence PQ makes an angle α with the horizontal.

[Attempted by about half the candidates, and well done by them.]

(ii)



From triangle BRS,

$\Rightarrow \text{BS} = \frac{h}{\tan\alpha}$

$\text{BS} = x_S - x_B$

$x_S = \frac{U\cos\alpha}{k}(1-e^{-kt})$

$x_B = \frac{u\cos\alpha}{k}(1-e^{-kt})$, then

$x_S = \frac{U}{u}x_B$

$\Rightarrow \text{BS} = \frac{U-u}{u}x_B$

Hence the horizontal displacement is $x_B = \frac{uh}{U-u}\cot\alpha$.

$$x_B = \frac{u\cos\alpha}{k}(1-e^{-kt})$$

$$\frac{uh}{U-u}\cot\alpha = \frac{u\cos\alpha}{k}(1-e^{-kt})$$

$$\frac{h}{(U-u)\sin\alpha} = \frac{1}{k}(1-e^{-kt})$$

$$\therefore \frac{h}{(U-u)\sin\alpha} = \frac{1}{k}(1-e^{-kt})$$

$$\frac{h}{(U-u)\sin\alpha} - 1 = -e^{-kt}$$ but $t = T$ at B

$$\therefore e^{-kT} = \frac{(U-u)\sin\alpha - kh}{(U-u)\sin\alpha} \qquad (*)$$

[Generally well answered by most who attempted it.]

(b) (iii) $\dot{x} = u \cos \alpha . e^{-kt}$ $\dot{y} = \frac{g + ku \sin \alpha}{k} e^{-kt} - \frac{g}{k}$ by differentiation of $\vec{r}(t)$

$\tan \theta = \frac{\dot{y}}{\dot{x}} = \tan \alpha + \frac{g}{ku \cos \alpha}(1 - e^{-kt})$

Replacing e^{-kt} from (*) and simplifying:

$\tan \theta = \tan \alpha + \frac{gh}{u \cos \alpha (kh - (U - u) \sin \alpha)}$

[As above. Many failed to simplify appropriately.]

(c) The motion is SHM, so $x = A\cos(nt+\alpha)+c$ (Reference sheet)

$$v = \dot{x} = -An\sin(nt+\alpha)$$

$$v^2 = A^2n^2\sin^2(nt+\alpha)$$

$$= n^2(A^2 - A^2\cos^2(nt+\alpha))$$

$$v^2 = n^2(A^2-(x-c)^2)$$

v will have stationary points where $a = 0$

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x-c)$$

So $a = 0$ when $x = c$, i.e. when $v^2 = n^2(A^2)$

Max velocity is $v = nA$ $-nA$ is extraneous as we seek max v

Noting $n = \frac{2\pi}{T}$, $v_{max} = \frac{2\pi A}{T}$

[Generally well answered, based on definitions, and the reference sheet.]

Q16) a) $P(\omega) = a\omega^4 + ib\omega^3 + c\omega^2 + id\omega + e = 0$

$$\overline{a\omega^4 + ib\omega^3 + c\omega^2 + id\omega + e} = \overline{0}$$

$$\overline{a\omega^4} + \overline{ib\omega^3} + \overline{c\omega^2} + \overline{id\omega} + \overline{e} = 0$$

$$\overline{a}\,\overline{\omega^4} + \overline{ib}\,\overline{\omega^3} + \overline{c}\,\overline{\omega^2} + \overline{id}\,\overline{\omega} + e = 0$$

$$a(\overline{\omega})^4 - ib(\overline{\omega})^3 + c(\overline{\omega})^2 - id(\overline{\omega}) + e = 0$$

$$a(-\overline{\omega})^4 + ib(-\overline{\omega})^3 + c(-\overline{\omega}) + id(-\overline{\omega}) + e = 0$$

$$\therefore P(-\overline{\omega}) = 0$$

COMMENT:

The conjugate root theorem cannot be applied because not all coefficients are real. It is basically the proof of the conjugate root theorem.

b) i) $\vec{a}, \vec{b}, \vec{c}$ form a triangle

$\therefore \vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $\vec{a}, \vec{b}, \vec{c} \neq \vec{0}$

$$\vec{u} + \vec{v} + \vec{w} = \cancel{(\vec{b}\cdot\vec{c})\vec{a}} - \cancel{(\vec{c}\cdot\vec{a})\vec{b}} + \cancel{(\vec{a}\cdot\vec{c})\vec{b}} - \cancel{(\vec{a}\cdot\vec{b})\vec{c}} + \cancel{(\vec{b}\cdot\vec{a})\vec{c}} - \cancel{(\vec{c}\cdot\vec{b})\vec{a}}$$

$$= \vec{0}$$

$\therefore \vec{u}, \vec{v}, \vec{w}$ also forms a triangle

ii) $\vec{u}\cdot\vec{c} = \left[(\vec{b}\cdot\vec{c})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}\right]\cdot\vec{c}$

$$= (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{c}) - (\vec{c}\cdot\vec{a})(\vec{b}\cdot\vec{c})$$

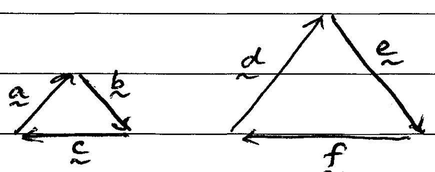
$$= 0$$

iii) $\vec{u}$ and $\vec{c}$ are perpendicular

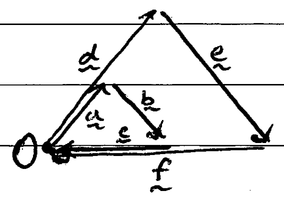
similarly $\vec{v}$ and $\vec{a}$ are perpendicular

$\vec{w}$ and $\vec{b}$ are perpendicular

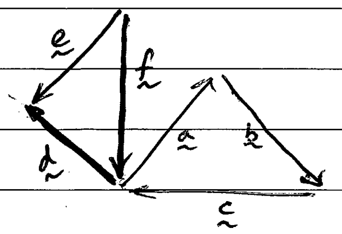
consider $\vec{d}, \vec{e}, \vec{f}$ which are in the same direction (respectively) as $\vec{a}, \vec{b}, \vec{c}$ and also form a triangle



If we make $\underset{\sim}{a}$ & $\underset{\sim}{d}$ position vectors its easy to see that those two triangles will be similar



Now, consider $\vec{d}, \vec{e}, \vec{f}$ which are a 90° rotation anticlockwise of $\vec{a}, \vec{b}, \vec{c}$ respectively.



rotations preserve congruency and so 90° anticlockwise for each would also be fine.

What if $\vec{a}, \vec{b}$ were anticlockwise while $\vec{c}$ was clockwise?

we would have vectors $\vec{d}, \vec{e}, -\vec{f}$

Assume $\vec{d}, \vec{e}, -\vec{f}$ also forms a triangle

ie $\vec{d} + \vec{e} + -\vec{f} = \vec{0}$ ——— (1)

$\vec{d} + \vec{e} + \vec{f} = \vec{0}$ ——— (2)

(2) $-$ (1)

$\vec{d} + \vec{e} + \vec{f} - (\vec{d} + \vec{e} - \vec{f}) = \vec{0}$

$2\vec{f} = \vec{0}$

$\vec{f} = \vec{0}$

A contradiction $\therefore$ can't form a triangle

$\vec{u}, \vec{v}, \vec{w}$ are perpendicular to $\vec{c}, \vec{a}, \vec{b}$ respectively
Since $\vec{u}, \vec{v}, \vec{w}$ form a triangle
they must be a rotation of 90° in the same direction and thus the triangles formed by $\vec{a}, \vec{b}, \vec{c}$ and $\vec{u}, \vec{v}, \vec{w}$ will be similar.

COMMENT:

The sum of two (or more) vectors will be a vector not a scalar. Students received $1\frac{1}{2}$ marks if they proved $\vec{u}+\vec{v}+\vec{w}=0$ instead of $\vec{u}+\vec{v}+\vec{w}=\vec{0}$.

An answer of 0 in part (ii) without working did not receive full marks.

Students needed to account for the fact that there are two vectors which will be perpendicular to another to gain full marks in part (iii).

c) i) $I_n = \int_0^1 x^{2n+1} e^{-x^2}\,dx$

$= -\frac{1}{2}\int_0^1 x^{2n} . -2xe^{-x^2}\,dx$

$= -\frac{1}{2}\left[\left[x^{2n} . e^{-x^2}\right]_0^1 - \int_0^1 e^{-x^2} . 2nx^{2n-1}\,dx\right]$

$= -\frac{1}{2}\left[(1)^{2n}e^{-(1)^2} - \cancel{(0)^{2n}e^{(0)^2}} - 2n\int_0^1 x^{2n-1}e^{-x^2}\,dx\right]$

$= -\frac{1}{2}\left[\frac{1}{e} - 2n\int_0^1 x^{2(n-1)+1} e^{-x^2}\,dx\right]$

$= -\frac{1}{2e} + nI_{n-1}$

ii) $I_0 = \int_0^1 x^{2(0)+1} e^{-x^2}\,dx$

$= \int_0^1 xe^{-x^2}\,dx$

$= -\frac{1}{2}\int_0^1 -2xe^{-x^2}\,dx$

$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$

$= -\frac{1}{2}\left[e^{-(1)^2} - e^{-(0)^2}\right]$

$= -\frac{1}{2}\left[\frac{1}{e} - 1\right]$

$= \frac{1}{2} - \frac{1}{2e}$

iii) Prove true for $n=1$

$LHS = 1 + \frac{1}{1!}$

$= 2$

$RHS = e - \frac{2eI_1}{1!}$

$= e - 2eI_1$

$= e - 2e\left(-\frac{1}{2e} + 1.I_0\right)$

$= e + 1 - 2eI_0$

$= e + 1 - 2e\left(\frac{1}{2} - \frac{1}{2e}\right)$

$= e + 1 - e + 1$

$= 2$

$LHS = RHS$ $\therefore$ true for $n=1$

Assume true for $n=k$, $k \in \mathbb{Z}^+$

$1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!} = e - \frac{2eI_k}{k!}$

Prove true for $n=k+1$

ie $1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!} + \frac{1}{(k+1)!} = e - \frac{2eI_{k+1}}{(k+1)!}$

$LHS = \underbrace{1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!}} + \frac{1}{(k+1)!}$

$= e - \frac{2eI_k}{k!} + \frac{1}{(k+1)!}$

$= e - \frac{2e(k+1)I_k}{(k+1)!} + \frac{1}{(k+1)!}$

$= e - \frac{2e\left((k+1)I_k - \frac{1}{2e}\right)}{(k+1)!}$

$= e - \frac{2eI_{k+1}}{(k+1)!}$ using (i)

$= RHS$ $\therefore$ true for $n=k+1$

$\therefore$ true by induction for positive integers n.

iv) $0 \le x \le 1$

$0^{2n+1} \le x^{2n+1} \le 1^{2n+1}$ since $f(x) = x^{2n+1}$ is an increasing function

$0 \le x^{2n+1} \le 1$

$0 \le x \le 1$

$0 \le x^2 \le 1$

$0 \geq -x^2 \geq -1$ since $f(x) = -x$ is a decreasing function

$-1 \le -x^2 \le 0$

$e^{-1} \le e^{-x^2} \le e^0$ since $f(x) = e^x$ is an increasing function

$\frac{1}{e} \le e^{-x^2} \le 1$

$\therefore 0 \le e^{-x^2} \le 1$

$\therefore 0 \le x^{2n+1} e^{-x^2} \le 1$

$$\int_0^1 0 \, dx \le \int_0^1 x^{2n+1} . e^{-x^2} \, dx \le \int_0^1 1 . dx$$

$$\underline{0 \le I_n \le 1}$$

v) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} = e - \frac{2e I_n}{n!}$

as $n \to \infty$

$\frac{1}{n!} \to 0$

$\underline{\text{ie}}$ $\frac{2e I_n}{n!} \to 0$ as I_n remains between 0 and 1.

$$\lim_{n \to \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \right) = \lim_{n \to \infty} \left(e - \frac{2e I_n}{n!} \right)$$

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + 0 = e - 0$

$\therefore e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$

COMMENT:

In part (i) students need to be showing more working for a show that question.
Show the substitution of limits and why $2n-1 = 2(n-1)+1$.
Students could have been penalised for this.
In part (iii) many students didn't prove true for the case $n=1$ correctly. Proofs should be clear and easy to follow.
In part (v) $I_n \to 0$ as $n \to \infty$. However, that argument has not been made.